galeDuality v1.0 Manual

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galeDuality

1 Introduction to galeDuality

The program galeDuality by Dan Bates, Jonathan Hauenstein, Matt Niemerg, and Frank Sottile implements algorithms based on [1], [3]. This manual provides information on how to use the software while the aforementioned articles provide more of the mathematical theory behind the algorithms.

2 Compiling galeDuality

galeDuality is written in C++ and requires the use of of three open-source software packages in order to run correctly. These packages are GAP [4], Bertini [2], and alphaCertified [5]. See the corresponding websites for each package maintained by their respective authors as to how to properly install the software. The galeDuality software has the capacity to run the Khovanskii-Rolle Continuation scripts (included in this tarbell) for systems in which the number of monomials exceeds the number of variables by only 2. This functionality requires the use of the proprietary computer algebra system software known as Maple. The user does not have to use the Khovanskii-Rolle Continuation maple scripts, unless desired. The complete symbolic transformation can be performed without running these scripts.

After the installation of each of the software packages, they all must be placed in a directory of the $PATH environment variable, and the gap executable must be named "gap". In addition, an environment variable named "GALEDIR" must be added to find the maple subroutines and is also used in the subroutines themselves. To add this environment variable, in your $HOME directory, open the hidden file ".bash_profile" using your favorite text editor. For BASH shell users (the default for most users will be BASH), add the line, "export GALEDIR=`mydirectoryhere`", save the file and close the text editor, and restart your shell. `mydirectoryhere` is the absolute location of the directory that the maple subroutines are located. For users who are unsure, you can type "ps -p $$" and inspect the output. If the user is using a different shell, edit the appropriate shell start up file and use the appropriate export command for that shell.

The location of the directory can be anywhere, but for non-advanced users, the location of the 'include' directory in the *.tar file will suffice. If you change the location of these files, YOU MUST CHANGE THE ENVIRONMENT VARIABLE AGAIN!

A gnu MAKEFILE is included and should require no additional modifications, assuming that the precautions in this paragraph are adhered to.

After 'untarring' the tarbell, the user can make the program by moving into the source directory and then type 'make galeDuality' in the command line.

3 Using galeDuality

To use galeDuality, the user must create an appropriate input file that corresponds to either a fewnomial system or a gale system. There are no additional configuration settings the user needs to provide.

3.1 Input for Fewnomial Systems

The first line of the file indicates the excess of the number of monomials over the number of variables, i.e. if the number of monomials is 9 and the number of variables is 4, then number placed on this line is 5. The second line indicates the dimensionality of the components that correspond to the solutions of the fewnomial system. Currently, the software only works for the zero-dimensional case. The third line indicates the number of variables in the fewnomial system.

The next line is a list of the monomials, delimited by spaces. Exponentiation is indicated by the "$\wedge$". In addition, if an exponent is negative, it must have encapsulated on either side by either a pair of "()" or a pair of "{}". Multiplication of variables is indicated by the "*" character. Concatenation does not suffice. For example, if the monomial is $a^2b^{-1}c^4$, the corresponding input for this monomial would be either "$a \wedge 2 * b \wedge \{-1\} * c \wedge 4$" or "$a \wedge 2 * b \wedge (-1) * c \wedge 4$". The constant monomial is written as 1.

The next lines are the leading coefficients of each of the monomials in the fewnomial system, written as rationals. For example, if one of the functions in the fewnomial system is

$$f(a,b,c,d,e) = a^2b^3c^{-4}e^{-2} + \frac{1}{3}a^{-1}b^2d^{-3} + \frac{2}{5}d^2e^{-5}$$

then the leading coefficients of 1, $\frac{1}{3}$, and $\frac{2}{5}$ would appear in the appropriate column that corresponds to the monomials $a^2b^3c^{-4}e^{-2}, a^{-1}b^2d^{-3}$, and d^2e^{-5}, respectively.

Suppose we have the following fewnomial system.

$$\begin{aligned}
f_1(a,b,c,d,e) &= a^{-1}b^2c^2d - \frac{1}{2}b^2c - 2b^{-4}c^{-7}d^{-5}e^{-1} + 1\\
f_2(a,b,c,d,e) &= ac - \frac{1}{8}b^2c + \frac{1}{2}b^2c - 2b^{-4}c^{-7}d^{-5}e^{-1} - \frac{1}{2}\\
f_3(a,b,c,d,e) &= bc^4d^4 + \frac{1}{16}b^2c + \frac{3}{4}b^2c - 2b^{-4}c^{-7}d^{-5}e^{-1} - \frac{3}{2}\\
f_4(a,b,c,d,e) &= d + \frac{3}{8}b^2c + b^2c - 2b^{-4}c^{-7}d^{-5}e^{-1} - 4\\
f_5(a,b,c,d,e) &= e + \frac{1}{2}b^2c - 2b^2c - 2b^{-4}c^{-7}d^{-5}e^{-1} - 3
\end{aligned}$$

For the appropriate input file of this system, see the file "heptagon_102.dat" located in the examples folder.

3.2 Input for Gale Systems

The first line of this file indicates the number of variables (call this v) and the second line indicates the number of linears of a gale system (call this l). The next v lines are some basis of the null space of the monomial support matrix of the appropriate dual of this gale system. The monomial support need not necessarily be known. Essentially, there will be v lines, delimited by spaces, with l coordinates. The next line is a space delimited list of the variables followed by a 1 (to indicate the constant monomial). The next lines are the leading coefficients of each of the terms in the appropriate linears that form the positive chamber of the hyperplane arrangement of the gale system written as rationals.

Suppose we have the follow gale system.

$$\begin{aligned}
m_1(s_0,s_1) &= \left(\frac{-10}{11}s_0 + \frac{30}{11}s_1 - \frac{10}{11}\right)\left(s_0 - s_1 - \frac{1}{2}\right)^3 s_0 - (s_0+s_1-1)s_1^3\\
m_1(s_0,s_1) &= (s_0+s_1-1)^3 s_0^2 - \left(\frac{-10}{11}s_0 + \frac{30}{11}s_1 - \frac{10}{11}\right)^2\left(s_0 - s_1 - \frac{1}{2}\right)s_1^2
\end{aligned}$$

For the appropriate input file of this system, see the file "pentagon_20" located in the examples folder.

3.3 Running galeDuality

galeDuality is a menu-driven program requiring either options inputted by the user for various choices or an appropriate file that can be redirected as input to the compiled C++ program that mimic input that would normally be provided by the user. The first choice is to indicate the type of input file, either a fewnomial or a gale system. The second choice is whether or not to display the solutions of the dual or the symbolic transformation of the dual itself.

The third choice determines whether or not to use Khovanskii-Rolle to solve the gale system or to use standard homotopy continuation methods via Bertini to solve the appropriate dual and will only be prompted if the previous choice of the user was a display of the solutions to the dual. The next option is whether or not to use the parametrization of the monomials with or without the heuristic for choosing a 'good' parametrization. The last option the user provides indicates whether or not to use the basis vectors generated by GAP or to use the heuristic for the 'best' sign patterns.

If the solutions are to be displayed, galeDuality will ask the user the max number of times to refine the solutions using the Newton Iteration algorithm provided in alphaCertified and will perform up to this number of Newton Iterations until the system is certified using floating point soft certification initially. If the solutions are all certified using floating point operations, galeDuality will ask the user whether or not to perform a rational hard certification of the soft certified solutions.

4 Output of galeDuality

galeDuality does not write any files to the disk, but will rather display the appropriate solutions requested or display the appropriate dual of the input system. Depending on various choices provided by the user, galeDuality will display on the console, the output from the Maple script and alphaCertified. If the solutions are displayed, alphaCertified will produce its appropriate output files and one may find the user's choice of solutions to be displayed saved in the refinedPoints file (see the alphaCertified documentation for additional details of files generated by alphaCertified). The additional coordinates of the solutions found in this file can be safely ignored, as these coordinates correspond to changing the laurent system to an appropriate polynomial system with the same solutions, in the case the user chooses to display the fewnomial solutions. In the situation where the user chooses to display the solutions to the gale system, the first n coordinates may be safely ignored as these correspond to the solutions of the linears and not to the variables of the gale system.

References

[1] Daniel Bates and Frank Sottile. Khovanskii–rolle continuation for real solutions. *Foundations of Computational Mathematics*, 11(5):563–587, 2011.

[2] Daniel J. Bates, Jonathan D. Hauenstein, Andrew J. Sommese, and Charles W. Wampler. Bertini.

[3] Frédéric Bihan and Frank Sottile. Gale duality for complete intersections. In *Annales de l'institut Fourier*, pages 877–892. Chartres: L'Institut, 1950-, 2008.

[4] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.6.5*, 2013.

[5] Jonathan D. Hauenstein and Frank Sottile. alphacertified: Software for certifying numerical solutions to polynomial equations. Available at http://www.math.tamu.edu/∼sottile/research/stories/alphaCertified.